Robust Tracking Control of a Nonholonomic Mobile Robot in the Presence of Disturbances

Nobutaka WADA∗†, Masami SAEKI∗ and Haijun CHEN∗

Key Words: Moving Robot, Nonlinear Control, Robust Control, Unknown Disturbance

February 20, 2003

* Dept. of Mechanical System Engineering, Hiroshima University
1-4-1 Kagamiyama, Higashi-Hiroshima, Hiroshima, 739-8527, Japan

† TEL: +81-824-24-7585; FAX: +81-824-22-7193
E-mail: nwada@mec.hiroshima-u.ac.jp
Abstract

In this paper, we consider the tracking control problem of a nonholonomic mobile robot in the presence of bounded disturbances. We propose a method for modifying the control law of Jiang et al. (1997) so that the resulting control law guarantees global stability and the tracking error converges to a neighborhood about zero that can be made arbitrary small in the presence of disturbances. Further, we show that the tracking error converge to zero in the absence of disturbances. A numerical example is provided to show effectiveness of the method.
1 Introduction

Several methodologies for tracking control of mobile robots with nonholonomic constraints have been developed (see e.g., [4, 6, 7]). The common features of the methods are: 1) the time derivative of the Lyapunov function used to derive the nonlinear control law is negative semi-definite and asymptotic stability is shown by using Barbálat’s lemma [8], 2) the methods only deal with the cases where disturbances do not exist. There exist several ways for modifying the nonlinear control law so that the boundedness of the state trajectories is guaranteed against disturbances [8, 9]. However, these methods only treat the cases where the time derivative of the Lyapunov function is negative definite. Hence it is unclear whether the methods are effective to the case where the time derivative of the Lyapunov function is only negative semi-definite or not. On the other hand, in [3], it is shown that exploiting the damped dynamic oscillator makes it possible to construct a Lyapunov function whose time derivative is negative definite. However, the existence of the damped dynamic oscillator prohibits that the tracking errors converge to zero in the absence of disturbances and tends to deteriorate the initial response.

In this paper, we propose a method for modifying the control law of [6] so that the resulting control law guarantees global stability and the tracking error converges to a neighborhood about zero that can be made arbitrary small in the presence of bounded disturbances. Further, we show that the control law guarantees that the tracking error converge to zero in the absence of the disturbances.

This paper is organized as follows. In Section 2, we formulate the tracking control problem of the mobile robot. In Section 3, we show a control method in the case where the bounded disturbances exist. In section 4, we provide a numerical example to show effectiveness of the method. Finally, in Section 5, we summarize the paper.

Notations: A signal $v(t)$ defined on $[0, \infty)$ is said to belong to $\mathcal{L}_\infty$, i.e., $v \in \mathcal{L}_\infty$, if $\|v\|_{\mathcal{L}_\infty} \triangleq \sup_{t \in [0, \infty)} \|v\|_2 < \infty$, where $\|v\|_2 \triangleq (v^Tv)^{1/2}$. The asymptotic $\mathcal{L}_\infty$ norm of $v(t)$ is defined by $\|v\|_a \triangleq \lim_{t \to \infty} \sup \|v\|_2$. A signal $v(t)$ defined on $[0, \infty)$ is said to belong to $\mathcal{L}_1$, i.e., $v \in \mathcal{L}_1$, if $\|v\|_{\mathcal{L}_1} \triangleq \int_0^\infty \|v\|_2 dt < \infty$.

2 Problem formulation

Let us consider a wheeled mobile robot depicted in Fig.1. We assume that the dynamics of the robot can be described by the following differential equations.

$$M(q)\ddot{q} + G(q) + F(\dot{q}) + \tau_d = B(q)\tau + A^T(q)\lambda$$

$$A(q)\dot{q} = 0$$

where

$$M(q) \triangleq \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix}, \quad B(q) \triangleq \frac{1}{r} \begin{bmatrix} \cos \theta & \cos \theta \\ \sin \theta & \sin \theta \\ b & -b \end{bmatrix}$$

and

$$A(q) \triangleq \begin{bmatrix} \sin \theta - \cos \theta \theta \end{bmatrix}$$

$$J \in \mathbb{R}^{3 \times 3}, \quad m \in \mathbb{R}, \quad J \in \mathbb{R}^{3 \times 3}, \quad G(q) \in \mathbb{R}^{3 \times 1}, \quad F(\dot{q}) \in \mathbb{R}^{3 \times 1}, \quad B(q) \in \mathbb{R}^{3 \times 2}, \quad A(q) \in \mathbb{R}^{1 \times 3}$$

and $m$ denotes the mass of the robot, $J$ denotes the moment of inertia of the robot about a vertical axis through $P$, $q \triangleq [x, y, \theta]^T$, $\tau \triangleq [\tau_x, \tau_y]^T$ denotes the input vector, $\lambda \in \mathbb{R}^n$ is the vector of constraint forces, and $\tau_d \in \mathbb{R}^3$ denotes a disturbance. $2b$ is the width of the mobile robot and $r$ is the radius of the wheel. $M(q) \in \mathbb{R}^{3 \times 3}$ is the inertia matrix, $G(q) \in \mathbb{R}^{3 \times 1}$ is the gravity force, $F(\dot{q}) \in \mathbb{R}^{3 \times 1}$ is the surface friction, $B(q) \in \mathbb{R}^{3 \times 2}$ is the input transformation matrix, $A(q) \in \mathbb{R}^{1 \times 3}$ is the matrix associated with the nonholonomic constraints. We make the following assumption.
Assumption 1 There exists a positive scalar $\eta$ such that $\eta \geq \|d\|_{\infty}$.

Then we introduce a full rank matrix $S(q) \in \mathbb{R}^{3 \times 2}$ which is made up by the smooth vectors fields and satisfies

$$A(q)S(q) = 0 \quad (5)$$

It is straightforward to verify that the following matrix satisfies Eq.(5).

$$S(q) \triangleq \begin{bmatrix} \cos \theta & 0 \\ \sin \theta & 0 \\ 0 & 1 \end{bmatrix} \quad (6)$$

From Eqs.(2) and (5), a kinematic model of the mobile robot can be obtained as

$$\dot{q} = S(q)\nu \quad (7)$$

where $\nu(t) \triangleq [v, \omega]^T \in \mathbb{R}^2$.

Then, by differentiating Eq.(7), substituting the expression for $\ddot{q}$ into Eq.(1) and premultiplying by $S^T$, we have

$$\ddot{M} \dot{\nu} + \ddot{V}_m \nu + \ddot{G} + \ddot{F} + \ddot{d} = \ddot{B}\tau \quad (8)$$

where $\ddot{M} \triangleq S^T M S$, $\ddot{V}_m \triangleq S^T \dddot{S}$, $\dddot{G} \triangleq S^T \dddot{G}$, $\dddot{F} \triangleq S^T \dddot{F}$, $\dddot{B} \triangleq S^T \dddot{B}$, $\dddot{d} \triangleq S^T \dddot{d}$. Then, by applying the following feedback control law,

$$\tau = \dddot{B}^{-1}[\dddot{M}u + \dddot{V}_m \nu + \dddot{G} + \dddot{F}] \quad (9)$$

Eq.(8) can be converted to

$$\dot{\nu} = u + \dddot{d} \quad (10)$$

where $\dddot{d} \triangleq -\dddot{M}^{-1}\dddot{d} \in \mathbb{R}^2$ and $u \in \mathbb{R}^2$ is a new input vector.

Then we introduce a reference robot described by

$$\dot{q}_r = S(q_r)\nu_r \quad (11)$$

where $q_r \triangleq [x_r, y_r, \theta_r]^T$, $\nu_r \triangleq [\nu_r, \omega_r]^T$, and the following error coordinates $[7]$,

$$\begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r - x \\ y_r - y \\ \theta_r - \theta \end{bmatrix} \quad (12)$$
From Eqs.(7), (10)–(12), the error dynamics can be derived as
\[
\dot{\tilde{x}} = f(\tilde{x}) + g(\tilde{x}) [u + d]
\]
where
\[
f(\tilde{x}) \triangleq \begin{bmatrix}
\omega y_e - v + v_r \cos \theta_e \\
-\omega x_e + v_r \sin \theta_e \\
\omega_r - \omega \\
0 \\
0
\end{bmatrix}, \quad g(\tilde{x}) \triangleq \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix}
\]
and \(\tilde{x} \triangleq [x_e, y_e, \theta_e, v, \omega]^T\).

In this paper, we consider the following problem.

**Problem 1** Consider the system (13). Find a feedback control law such that, for arbitrary initial tracking error, 1) all the state trajectories are globally bounded and the tracking errors \(|x_e|, |y_e|, |\theta_e|\) converge to a neighborhood about zero that can be made arbitrary small in the presence of \(d\), and 2) \(|x_e|, |y_e|, |\theta_e|\) converge to zero in the case where there exists finite time \(t_d\) such that \(d(t) = 0, \forall t \geq t_d\).

### 3 Tracking control of a mobile robot

In this section, we first introduce a tracking control law of [6] that can be applied to the case where the disturbances do not exist. Then, we show a method for modifying the control law for the case where the disturbances exist.

#### 3.1 Tracking control in the absence of disturbances

By introducing the following new variables
\[
\bar{x}_e \triangleq x_e - c_3 \omega y_e, \quad \bar{v} \triangleq v - \alpha v, \quad \bar{\omega} \triangleq \omega - \alpha \omega
\]
\[
\alpha v \triangleq c_4 \bar{x}_e + v_r \cos \theta_e - c_3 \omega y_e + c_3 \omega [\omega x_e - v_r \sin \theta_e]
\]
\[
\alpha \omega \triangleq \omega_r + \gamma y_e v_r \sin \frac{\theta_e}{\theta_e} + c_5 \gamma \theta_e
\]
where \(c_3, c_4, c_5, \gamma > 0\), the system (13) can be transformed into
\[
\dot{x} = F(x) + G(x)[u + d]
\]
where
\[
F(x) \triangleq \begin{bmatrix}
\omega y_e - c_4 \bar{x}_e - \bar{v} \\
-\omega x_e + v_r \sin \theta_e \\
-c_5 \gamma \theta_e - \gamma y_e v_r \sin \frac{\theta_e}{\theta_e} - \bar{\omega} \\
-\dot{\bar{v}} \\
-\dot{\bar{\omega}}
\end{bmatrix}, \quad G(x) \triangleq \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix}
\]
and \(x(t) \triangleq [\bar{x}_e, y_e, \theta_e, \bar{v}, \bar{\omega}]^T\). Then we apply the following feedback control law
\[
u(x) = \Upsilon(x) \triangleq \begin{bmatrix}
\bar{x}_e + \dot{\bar{v}} - c_6 \bar{v} \\
\frac{1}{\gamma} \theta_e + \dot{\bar{\omega}} - c_7 \bar{\omega}
\end{bmatrix}
\]
where \( c_6, c_7 > 0 \), and consider the following Lyapunov function candidate.

\[
V(x) \triangleq \frac{1}{2} \dot{y}_e^2 + \frac{1}{2} \dot{\theta}_e^2 + \frac{1}{2} \dot{\omega}^2 + \frac{1}{2} \dot{\theta}_e^2 + \frac{1}{2} \dot{\omega}^2
\]  

(20)

Taking the time derivative of \( V \) along the solution trajectories of Eqs.(18) and (19) with \( d(t) = 0, \forall t \geq 0 \) yields

\[
\dot{V}(x) = -c_3 \omega^2 \dot{y}_e^2 - c_4 \dot{x}_e^2 - c_6 \dot{\theta}_e^2 - c_7 \dot{\omega}^2 \leq -W(x) \leq 0
\]  

(21)

Hence, in the case where \( d(t) = 0, \forall t \geq 0 \), the control law (19) guarantees global stability in the sense of Lyapunov. Furthermore, although \( W(x) \) is only positive semi-definite, it is shown in [6] that the control law also guarantees global asymptotic stability by applying Barbálat’s lemma.[8].

**Proposition 1** [6] Consider the system (13),(19) with \( d(t) = 0, \forall t \geq 0 \). Assume that \( v_r, \dot{v}_r, \omega_r, \dot{\omega}_r \) and \( \ddot{v}_r, \ddot{\omega}_r \) are bounded on \([0, \infty)\). Then all the state trajectories of the system (13), (19) are globally bounded. Furthermore, if \( v_r(t) \) does not converge to zero, or if \( v_r(t) \) tends to zero but \( \omega_r(t) \) does not converge to zero, then

\[
\lim_{t \to \infty} ||x_e|| + ||\theta_e|| + ||\omega - \omega_r|| + |v - v_r| = 0
\]  

(22)

### 3.2 Tracking control in the presence of disturbances

In the case of \( d \neq 0 \), the control law (19) may not guarantee not only the asymptotic property (22) but also boundedness of the state trajectory. To consider the case where the disturbance exists, we modify the control law (19) as follows.

\[
u(x) = Y(x) - \xi(x)
\]  

(23)

where

\[
\xi(x) \triangleq \begin{bmatrix} \eta \tanh(\kappa z_1) \\ \eta \tanh(\kappa z_2) \end{bmatrix}, \quad z \triangleq \begin{pmatrix} \partial V \\ \partial x \end{pmatrix}^T
\]

and \( \kappa \) is a positive scalar.

By taking the time derivative of \( V(x) \) along the solution trajectories of the system (18), (23), we have

\[
\dot{V} \leq -W(x) + \frac{2\eta}{\kappa}, \quad \forall x \in \mathcal{R}^5
\]  

(24)

The derivation of Eq.(24) is shown in Appendix A.

Since \( \dot{V} < 0, \forall x \in \Omega_W \triangleq \{ x \in \mathcal{R}^5 | W(x) > 2\eta/\kappa \} \), \( V \) monotonically decreases while \( x \in \Omega_W \), which implies that any trajectory \( x(t) \) starting from \( \Omega_W \) is bounded while \( x \in \Omega_W \). Then \( x(t) \) enters \( \Omega_W \triangleq \{ x \in \mathcal{R}^5 | W(x) \leq 2\eta/\kappa \} \) at some time or remains in \( \Omega_W, \forall t \geq 0 \). In the following, we first examine the behavior of \( x(t) \) after its arrival at \( \Omega_W \) in the case where \( x(t) \) enters \( \Omega_W \) (Case 1). Then we consider the case where \( x(t) \) remains in \( \Omega_W, \forall t \geq 0 \) (Case 2). Finally, we consider the case where the initial state value belongs to \( \Omega_W \), namely, \( x(0) \in \Omega_W \) (Case 3).

- **Case 1**: \( x(t) \) enters \( \Omega_W \)

  When \( x(t) \) enters \( \Omega_W \), the following inequality holds.

\[
c_3 \omega^2 \dot{y}_e^2 + c_4 \dot{x}_e^2 + c_6 \dot{\theta}_e^2 + c_7 \dot{\omega}^2 \leq \frac{2\eta}{\kappa}, \quad x \in \Omega_W
\]  

(25)
Inequality (25) implies that
\[
|\omega y_e| \leq \sqrt{\frac{2\eta}{c_5\kappa}} |x_e| \leq \sqrt{\frac{2\eta}{c_4\kappa}}, \quad |\theta_e| \leq \sqrt{\frac{2\eta}{c_6\kappa}}
\]
\[|v| \leq \sqrt{\frac{2\eta}{c_6\kappa}} \quad |\bar{\omega}| \leq \sqrt{\frac{2\eta}{c_7\kappa}}, \quad \forall x \in \Omega_W
\] (26)

From (26), we can conclude that the state variables except for \(y_e\) are bounded on \(\Omega_W\). Then we examine the boundedness of \(y_e(t)\) on \(\Omega_W\). In the following, we assume that there exists a constant \(v_{r\text{min}}\) such that \(0 < v_{r\text{min}} \leq v_r, \forall t \geq 0\). From Eqs.(15) and (17), we have
\[
\omega = A y_e + B
\] (27)
where \(A(t) \triangleq \gamma v_r \sin \frac{\theta}{\eta}, \quad B(t) \triangleq \bar{\omega} + \omega_r + c_5 \gamma \theta_e\). Then, by substituting (27) for the 1st inequality of (26), we have
\[
|\omega y_e| = |(A y_e + B) y_e| \leq \rho, \quad \forall x \in \Omega_W
\] (28)
where \(\rho \triangleq \sqrt{2\eta/c_3\kappa}\). If we choose \(c_5\) and \(\kappa\) such that \(\sqrt{2\eta/c_3\kappa} < \pi\), then \(A(t) > 0, \forall x \in \Omega_W\). In this case, from the inequality (28), the upper bound of \(|y_e|\) can be obtained as
\[
|y_e| \leq \frac{1}{2} \left( \frac{|B|}{A} + \sqrt{\left( \frac{|B|}{A} \right)^2 + 4 \rho} \right)
\]
\[
\leq \frac{1}{2} \left( \Phi + \sqrt{\Phi^2 + 4 \rho \Psi} \right), \quad \forall x \in \Omega_W
\] (29)
where
\[
\Psi \triangleq \frac{1}{\gamma v_{r\text{min}}} \sin \left( \sqrt{\frac{2\eta}{c_5\kappa}} \right)
\]
\[
\Phi \triangleq \Psi(\gamma, \epsilon) \left[ \omega_{r\text{max}} + \sqrt{\frac{2\eta}{\kappa}} \left( \frac{1}{\sqrt{c_7}} + \gamma \sqrt{c_5} \right) \right]
\]
and \(\omega_{r\text{max}}\) is a constant such that \(\omega_{r\text{max}} \geq |\omega_r|, \forall t \geq 0\). Note that \(\Psi(\gamma, \epsilon)\) and \(\Phi(\gamma, \epsilon)\) can be made arbitrary small by choosing large values as \(\gamma\) and \(\kappa\).

From Eq.(29), when \(v_r \geq v_{r\text{min}} > 0, \forall t \geq 0\) and \(\sqrt{2\eta/c_3\kappa} < \pi\), \(|y_e|\) is also bounded on \(\Omega_W\). Therefore, \(x(t)\) is bounded whenever \(x(t) \in \Omega_W\) since \(\Omega_W \in \Omega_x\), where \(\Omega_x\) is a bounded set defined by \(\Omega_x \triangleq \{x \in \mathcal{R}^5|\text{from the 2nd to the 5th inequalities of (26), (29)}\}\) (see Fig.2).

Since \(\Omega_W\) is not an invariant set in general, \(x(t)\) may leave \(\Omega_W\) at some time. Hence, in the following, we consider the behavior of \(x(t)\) after it leaves \(\Omega_W\). To consider the case, we define a level set \(L_V(\beta) \triangleq \{x \in \mathcal{R}^5|V(x) \leq \beta\}\) such that \(\Omega_x \in L_V(\beta)\). Such a positive scalar \(\beta\) can be derived by substituting the right side of Eqs.(26) and (29) for Eq.(20) as follows.
\[
\beta \triangleq \frac{\eta}{c_4\kappa} + \frac{1}{8} \left( \Phi + \sqrt{\Phi^2 + 4 \rho \Psi} \right)^2 + \frac{\eta}{c_6\kappa} + \frac{\eta}{c_7\kappa}
\] (30)
The relationship between \(\Omega_x\) and \(L_V(\beta)\) is shown in Fig.2. Note that \(\beta\) defined in Eq.(30) can be made arbitrary small by choosing large values as \(\gamma\) and \(\kappa\). Since \(\Omega_x \in L_V(\beta)\) and \(\dot{V} < 0\) outside of \(\Omega_W\), \(x(t)\) moves inside \(L_V(\beta)\) and approaches \(\Omega_W\) again. After that, although \(x(t)\) may reach and leave \(\Omega_W\) repeatedly, \(x(t)\) stays in \(L_V(\beta)\).
Figure 2: Trajectory of the state variable $x(t)$

- Case 2: $x(t)$ remains in $\bar{\Omega}_W$, $\forall t \geq 0$
  
  When $x(t) \in \bar{\Omega}_W$, $\forall t \geq 0$, $V$ monotonically decrease toward a constant $c > 0$ since $\dot{V} < 0$ and $x = 0 \in \Omega_W$. In this case, $\dot{V}$ converges to zero as $t \to \infty$, which implies that $x(t)$ is globally bounded and converges to the boundary of $\Omega_W$ as $t \to \infty$. This implies that $x(t)$ converges to $L_V(\beta)$ as $t \to \infty$ since $\Omega_W \in L_V(\beta)$.

- Case 3: $x(0) \in \Omega_W$
  
  Since $\Omega_W \in L_V(\beta)$, $x(0) \in \Omega_W$ implies $x(0) \in L_V(\beta)$. Hence, in this case, it is clear that $x(t) \in L_V(\beta), \forall t \geq 0$.

From the above discussion, we can conclude that any trajectory $x(t)$ converges to the level set $L_V(\beta)$ as $t \to \infty$. Then, based on the result, we show that the upper bound of the asymptotic $L_\infty$ norm of the error signals $x_e(t), y_e(t), \theta_e(t)$ can be made arbitrary small by appropriately choosing the parameters of the control law. Since any state trajectory $x(t)$ converges to $L_V(\beta)$ as $t \to \infty$, the following inequalities holds.

$$
\|\bar{x}_e\|_a \leq \sqrt{2}\beta, \quad \|y_e\|_a \leq \sqrt{2}\beta, \quad \|\theta_e\|_a \leq \sqrt{2}\beta\gamma \\
\|\bar{v}\|_a \leq \sqrt{2}\beta, \quad \|\bar{w}\|_a \leq \sqrt{2}\beta 
$$

(31)

Therefore, by choosing a sufficiently small value as $\beta$, $\|y_e\|_a$ can be made arbitrary small. Next we consider the upper bound of $\|x_e\|_a$. From Eqs. (15), (17) and (31), we have

$$
\|x_e\|_a \leq \|\bar{x}_e\|_a + c_3\|\bar{w}\|_a\|y_e\|_a \\
\leq \sqrt{2}\beta + c_3\sqrt{2}\beta(\sqrt{2}\beta + \|\bar{w}\|_a + \gamma\sqrt{2}\beta\|v_r\|_a + c_5\gamma\sqrt{2}\beta\gamma) \\
(32)
$$

Hence, by choosing sufficiently small values as $\beta$ and $c_3$, the upper bound of $\|x_e\|_a$ can be made arbitrary small. (Although the right side of Eq. (32) becomes large when we choose a large value as $\gamma$, it can be made small by choosing a sufficiently small value as $c_3$.) Finally, we consider the upper bound of $\|\theta_e\|_a$. From Eq. (30), the following equality can be derived.

$$
\beta\gamma = \frac{\gamma\eta}{c_4\kappa} + \frac{1}{8}\left(\sqrt{\gamma\Phi} + \sqrt{\gamma\Phi^2 + 4\rho\gamma^2}\right)^2 + \frac{\eta}{c_5\kappa} + \frac{\gamma\eta}{c_6\kappa} + \frac{\gamma\eta}{c_7\kappa} \\
(33)
$$

From Eq. (33), it can be verified that the value $\beta\gamma$ can be made arbitrary small by choosing a sufficiently large value as $\kappa$, which implies that the upper bound of $\|\theta_e\|_a$ can be made arbitrary small.
Theorem 1  Consider the system (18),(23). Assume that Assumption 1 is satisfied. Further, assume that 1) there exist constants $x_{r_{min}}$ and $\omega_{r_{max}}$ such that $0 < x_{r_{min}} \leq x_r, \forall t \geq 0$ and $|\omega_r| \leq \omega_{r_{max}}, \forall t \geq 0$, 2) $v_r, \bar{v}_r, \omega_r, \omega_r$ are bounded on $[0, \infty)$ and 3) the design parameters $\kappa$ and $c_5$ are chosen such that $\sqrt{2|\eta/c_5\kappa} < \pi$. Then all the state trajectories of the system (18),(23) are globally bounded. Moreover, the upper bound of $\|e_x\|_a, \|y_c\|_a, \|\theta_c\|_a$ can be made arbitrary small by choosing large values as $\gamma, \kappa$ and a small value as $c_3$.

Furthermore, assume that there exists finite time $t_d$ such that $\tau_d(t) = 0, \forall t \geq t_d$. Then, if $v_r(t)$ does not converge to zero, or if $v_r(t)$ tends to zero but $\omega_r(t)$ does not converge to zero, then

$$\lim_{t \to \infty} \|e_x\| + \|y_c\| + \|e_c\| + |\omega - \omega_r| + |v - v_r| = 0$$  \hspace{1cm} (34)

Proof) The first half of Theorem 1 has been already proven. The proof of the asymptotic property (34) is shown in Appendix B. □

Note 1 The boundedness of the state trajectory of the system (18),(23) is guaranteed in the case where the conditions of Theorem 1 are satisfied. By using the result, it can be verified that the original state trajectory, namely, $[x_e, y_e, \theta_c, v, \omega]^T$, is also bounded (see Appendix D). However, unfortunately, this does not always imply boundedness of the control input $u(x)$. In order to guarantee boundedness of $u(x)$, several additional conditions, namely, $\bar{v}_r, \bar{\omega}_r \in \mathbb{L}_\infty$ and $\tau_{d1}$, $\tau_{d2}$ are required (see Appendix E).

Note 2 The method of modifying the control law of this paper can be regarded as a so-called Lyapunov redesign technique [8, 9]. However, the usual Lyapunov redesign technique [8, 9] only treats the case where $W(x)$ is positive definite. Hence, since $W(x)$ of the control problem of this paper is positive semi-definite, the techniques used to prove stability in [8, 9] can not be applied to the problem of this paper. Moreover, as opposed to the usual Lyapunov redesign technique, the term $\xi(x)$ introduced in Eq.(23) has two important properties, namely, smoothness and boundedness. These properties enable us to prove that 1) $u(x)$ is bounded (see Appendix E) and 2) the asymptotic $\mathbb{L}_\infty$ norm of the signals $x_e, y_e, \theta_c$ can be made arbitrary small with keeping amplitude of $\xi(x)$ less than $\eta$.

4 Numerical Example

In this numerical example, physical parameters and design parameters are $m = 1, J = 1, r = 1, b = 1, G(q) = 0, \forall q \in \mathbb{R}^3, F(q) = 0, \forall q \in \mathbb{R}^3, c_3 = c_4 = c_5 = c_7 = 1, \gamma = 10, \kappa = 100$. The bounded disturbance is

$$\tau_d(t) = \begin{cases} \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} \cos \left( \frac{\pi}{3} t \right), & (0 \leq t \leq 3) \\ \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, & (3 \leq t) \end{cases} \hspace{1cm} (35)$$

The positive scalar $\eta$ is chosen as $\eta = 6$. The initial condition is $q_r(0) = [0, 0, 0]^T$, $q(0) = [-0.5, 0.5, -\frac{\pi}{4}]^T$.

Further, $v_r = 1, \omega_r = 0.5$.

In Figs.3-5, it is shown that the state trajectory of the real robot with $\xi(x)$ is quite similar to that of the reference robot. On the other hand, the state trajectory of the real robot without $\xi(x)$ is deteriorated by the disturbance.
5 Conclusion

In this paper, we have considered the tracking control problem of a nonholonomic mobile robot in the presence of bounded disturbances. We have shown that, by using the proposed control law, all the state variables are globally bounded and the tracking error converges to a neighborhood about zero that can be made arbitrary small in the presence of disturbances. Furthermore, we have shown that, by using the control law, the tracking error converges to zero in the case where there exists finite time $t_d$ such that $\tau_d(t) = 0, \forall t \geq t_d$.

References


A The time derivative of $V(x)$

The upper bound of time derivative of $V(x)$ along the solution trajectories of the system (18), (23) is given as follows.

\[
\dot{V} = \frac{\partial V}{\partial x} [F + G(u + d)] \\
= \frac{\partial V}{\partial x} [F + GT] + \frac{\partial V}{\partial x} G [-\xi + d] \\
\leq -W(x) + z^T [-\xi + d] \\
= -W(x) + [-\eta_1 \tanh(\kappa z_1) + z_1 d_1] + [-\eta_2 \tanh(\kappa z_2) + z_2 d_2] (36)
\]

Firstly, let us consider the upper bound of the 2nd term of the right side of Eq.(36).

In the case of $z_1 \geq 0$, the following inequality holds.

\[-\eta_1 \tanh(\kappa z_1) + z_1 d_1 \leq E_1(z_1), \forall z_1 \geq 0 (37)\]
where $\Xi^+_1(z_1) \triangleq -\eta_1 \tanh(\kappa z_1) + z_1 \eta$. By taking the derivative of $\Xi^+_1(z_1)$ on $z_1$, we have

$$\frac{d\Xi^+_1}{dz_1} = -\eta \left[ \tanh(\kappa z_1) + \frac{\kappa z_1}{\cosh(\kappa z_1)^2} - 1 \right]$$  \hspace{1cm} (38)

From Eq.(38), it can be shown that $d\Xi^+_1/dz_1 = 0$ iff

$$2\kappa z_1 - 1 = e^{-2\kappa z_1}$$  \hspace{1cm} (39)

It is clear that Eq.(39) has a unique solution $z_{10} > 0$. Then, taking the 2nd derivative of $\Xi^+_1$ on $z_1$ yields

$$\frac{d^2\Xi^+_1}{dz_1^2} = -\eta \left[ \frac{2\kappa}{\cosh(\kappa z_1)^2} + \frac{2\kappa^2 z_1 \sinh(\kappa z_1)}{\cosh(\kappa z_1)^3} \right] < 0, \ \forall z_1 \geq 0$$  \hspace{1cm} (40)

Eq.(40) implies that $\Xi^+_1$ is a concave function in $z_1 \geq 0$. Hence, $\Xi^+_1$ achieves the maximum value at $z_1 = z_{10}$. Therefore, the following inequalities hold.

$$\Xi^+_1(z_1) \leq \Xi^+_1(z_{10})$$

$$= \eta z_{10} \left[ 1 - \tanh(\kappa z_{10}) \right]$$

$$= \frac{2\eta z_{10} e^{-2\kappa z_{10}}}{1 + e^{-2\kappa z_{10}}}$$

$$= \frac{\eta}{\kappa} e^{-2\kappa z_{10}} \leq \frac{\eta}{\kappa}, \ \forall z_1 \geq 0$$  \hspace{1cm} (41)

To derive the 3rd equality, we used $2\kappa z_{10} - 1 = e^{-2\kappa z_{10}}$.

Then let us consider the case of $z_1 \leq 0$. In this case, the following inequality holds.

$$-\eta z_1 \tanh(\kappa z_1) + z_1 d_1 \leq \Xi^-_1(z_1), \ \forall z_1 \leq 0$$  \hspace{1cm} (42)

where $\Xi^-_1(z_1) \triangleq -\eta_1 \tanh(\kappa z_1) - z_1 \eta$. Then, based on the similar procedures in the case of $z_1 \geq 0$, we can show that

$$\Xi^-_1(z_1) \leq \frac{\eta}{\kappa}, \ \forall z_1 \leq 0$$  \hspace{1cm} (43)

Therefore, from Eqs.(41) and (43), we obtain

$$-\eta z_1 \tanh(\kappa z_1) + z_1 d_1 \leq \frac{\eta}{\kappa}, \ \forall z_1 \in \mathcal{R}$$  \hspace{1cm} (44)

Similarly, we can show that

$$-\eta z_2 \tanh(\kappa z_2) + z_2 d_2 \leq \frac{\eta}{\kappa}, \ \forall z_2 \in \mathcal{R}$$  \hspace{1cm} (45)

Hence, from Eqs.(36), (44), (45), we obtain

$$\dot{V} \leq -W(x) + \frac{2\eta}{\kappa}, \ \forall x \in \mathcal{R}^5$$  \hspace{1cm} (46)

B Proof of Eq.(34)

Since the control system considered in this paper is time invariant, in the following, we assume that $t_d = 0$. Clearly, $x(0)$ belongs to a bounded set. In the case where $\tau_d(t) = 0, \forall t \geq 0$, the time derivative of $V(x)$ along the solution trajectories of the system (18), (23) is given as follows.

$$\dot{V} \leq -W(x) - \eta z_1 \tanh(\kappa z_1) - \eta z_2 \tanh(\kappa z_2) \leq 0$$  \hspace{1cm} (47)
Hence, the closed-loop system (18), (23) is globally stable in the sense of Lyapunov and
\[ \dot{x}_e, y_e, \theta_e, \bar{v}, \bar{\omega} \in \mathcal{L}_\infty \] (48)

Then, from (17) and (48),
\[
|\alpha_\omega| \leq |\omega_r| + \gamma|y_e||v_r| \left| \frac{\sin \theta_e}{\theta_e} \right| + c_5 \gamma |\theta_e| \\
\leq \omega_{r \text{ max}} + \gamma\|y_e\|_{\mathcal{L}_\infty} v_{r \text{ max}} + c_5 \gamma \|\theta_e\|_{\mathcal{L}_\infty} 
\] (49)

Hence, from (15), (48) and (49), we can conclude that
\[
\alpha_\omega \in \mathcal{L}_\infty, \omega = \alpha_\omega + \bar{\omega} \in \mathcal{L}_\infty, x_e = \bar{x}_e + c_3 \omega y_e \in \mathcal{L}_\infty 
\] (50)

By the way, by integrating the both sides of (47) over \([0, t]\), we obtain
\[
0 < V(0) - c_3 \int_0^t \omega^2 y_e^2 dt - c_4 \int_0^t \dot{x}_e^2 dt - c_5 \int_0^t \theta^2_e dt + c_6 \int_0^t \dot{\bar{v}}^2 dt - c_7 \int_0^t \bar{\omega}^2 dt \\
- \eta \int_0^t z_1 \tanh(\kappa z_1) dt - \eta \int_0^t z_2 \tanh(\kappa z_2) dt 
\] (51)

Note that \(z_i \tanh(\kappa z_i) \geq 0, \forall i = 1, 2\). Therefore, since the inequality (51) still holds even if \(t \to \infty\), we get
\[
\omega^2 y_e^2, \bar{x}_e^2, \theta_e^2, \bar{v}^2, \bar{\omega}^2 \in \mathcal{L}_1 
\] (52)

Then we show (34) by using (48), (50), (52), Lemma 1 and Lemma 2.

- \(\theta_e^2\): From (18) and (48),
\[
\frac{d}{dt}(\theta_e^2) = 2 \theta_e \dot{\theta}_e \\
= 2 \theta_e \left( -c_5 \gamma \theta_e - \gamma y_e v_r \frac{\sin \theta_e}{\theta_e} - \bar{\omega} \right) \in \mathcal{L}_\infty 
\] (53)

Hence, with the help of Lemma 1, it follows that \(\theta_e \to 0, (t \to \infty)\).

- \(\bar{x}_e^2\): From (18), (48) and (50),
\[
\frac{d}{dt}(\bar{x}_e^2) = 2 \bar{x}_e \dot{\bar{x}}_e \\
= 2 \bar{x}_e \left( \omega y_e - c_4 \bar{x}_e - \bar{\bar{v}} \right) \in \mathcal{L}_\infty 
\] (54)

Hence, with the help of Lemma 1, it follows that \(\bar{x}_e \to 0, (t \to \infty)\).

- \(\omega^2 y_e^2\): From (18), (48) and (50),
\[
\dot{y}_e = -c_3 \omega^2 y_e - \omega \bar{x}_e v_r \sin \theta_e \in \mathcal{L}_\infty 
\] (55)

Thus,
\[
\dot{\alpha}_\omega = \dot{\omega}_r + \gamma y_e v_r \frac{\sin \theta_e}{\theta_e} + \gamma y_e \bar{v} \frac{\sin \theta_e}{\theta_e} + \gamma y_e v_r \frac{d}{dt} \left( \frac{\sin \theta_e}{\theta_e} \right) + c_5 \gamma \theta_e \in \mathcal{L}_\infty 
\] (56)

By the way, from the definition, \(\|\xi_2\|_{\mathcal{L}_\infty} \leq \eta\). Hence,
\[
\dot{\bar{\omega}} = \dot{\omega} + \dot{\alpha}_\omega \\
= (-\dot{\alpha}_\omega + u_2) + \dot{\alpha}_\omega \\
= \frac{1}{\gamma} \dot{\theta}_e + \dot{\alpha}_\omega - c_7 \bar{\omega} - \xi_2 \in \mathcal{L}_\infty 
\] (57)
Thus,
\[
\frac{d}{dt}(\omega^2 y_e^2) = 2\omega\dot{\omega} y_e^2 + 2\omega^2 y_e y_e \in \mathcal{L}_\infty
\]  
(58)

Hence, with the help of Lemma 1, it follows that \(\omega y_e \to 0, \ (t \to \infty)\).

\* \(x_e\): From (15),
\[
x_e = \bar{x}_e + c_5 \omega y_e \to 0, \ (t \to \infty)
\]  
(59)

\* \(y_e\):

- In the case where \(v_r\) tends to 0 but \(\omega_r\) does not converge to 0.

\[
\frac{d}{dt}(\bar{\omega}^2) = 2\bar{\omega}\dot{\bar{\omega}}
\]
\[
= 2(\omega - \alpha_\omega)(-\dot{\omega} + u_2)
\]
\[
= 2(\omega - \alpha_\omega)(\frac{1}{\gamma} \theta_e - c_7 \bar{\omega} + \xi_2) \in \mathcal{L}_\infty
\]  
(60)

Therefore, \(\bar{\omega} \to 0, \ (t \to \infty)\). Hence,
\[
\omega = \bar{\omega} + \alpha_\omega
\]
\[
= \bar{\omega} + (\omega_r + \gamma y_e v_r \frac{\sin \theta_e}{\theta_e} + c_5 \gamma \theta_e) \to \omega_r, \ (t \to \infty)
\]  
(61)

From the fact that \(\omega y_e \to 0, \ (t \to \infty)\), we can conclude that \(y_e \to 0, \ (t \to \infty)\).

- In the case where \(v_r\) does not converge to 0.

From (18),
\[
\dot{\theta}_e = -c_6 \gamma \theta_e + p(t)
\]  
(62)

where
\[
p(t) \equiv -\gamma y_e v_r \frac{\sin \theta_e}{\theta_e} - \bar{\bar{\omega}}
\]

It can easily be verified that \(p(t) \in \mathcal{L}_\infty\) and
\[
\dot{p}(t) = -\gamma y e v_r \frac{\sin \theta_e}{\theta_e} - \gamma y_e v_r \frac{\sin \theta_e}{\theta_e} - \gamma y e v_r \frac{d}{dt}(\frac{\sin \theta_e}{\theta_e}) - \dot{\bar{\bar{\omega}}} \in \mathcal{L}_\infty
\]  
(63)

Hence, from Lemma 2, \(p(t) \to 0, \ (t \to \infty)\). Since \(v_r(t)\) does not converge to 0, \(y_e(t) \to 0, \ (t \to \infty)\).

\* \(\omega - \omega_r\):

Firstly, we show that \(\bar{\omega}\) converges to 0 as \(t \to \infty\). From (18) and (23), we get
\[
\frac{d}{dt}\omega^2 = 2\omega\dot{\omega}
\]
\[
= 2\bar{\omega}(-\dot{\omega} + u_2)
\]
\[
= 2\bar{\omega}(\frac{1}{\gamma} \theta_e - c_7 \bar{\omega} + \xi_2) \in \mathcal{L}_\infty
\]  
(64)

Moreover, from (52), \(\bar{\omega}^2 \in \mathcal{L}_1\). Hence, with the help of Lemma 1, it follows that \(\bar{\omega} \to 0, \ (t \to \infty)\).

Therefore, since \(y_e \to 0, \ (t \to \infty)\), we can conclude that
\[
\omega - \omega_r = (\bar{\omega} + \alpha_\omega) - \omega_r
\]
\[
= \bar{\omega} + \gamma y e v_r \frac{\sin \theta_e}{\theta_e} + c_5 \gamma \theta_e \to 0, \ (t \to \infty)
\]  
(65)
In this section, we show that the original state trajectory, namely, \( \dot{\theta}_e, \dot{\omega}, \dot{\omega} \). From (18), we get

\[
\dot{\theta}_e = -c_5 \gamma \theta_e - \gamma y_e v_r \frac{\sin \theta_e}{\theta_e} - \ddot{\omega} \in \mathcal{L}_\infty \tag{66}
\]

From (66) and (17), we have

\[
\dot{\alpha}_\omega = \dot{\omega}_r + \gamma y_e v_r \frac{\sin \theta_e}{\theta_e} + \gamma y_e v_r \frac{\sin \theta_e}{\theta_e} + \gamma y_e v_r \frac{d}{dt} \frac{\sin \theta_e}{\theta_e} + c_5 \gamma \dot{\theta}_e \in \mathcal{L}_\infty \tag{67}
\]

Here, we used the fact that \( \frac{d}{dt} \left( \frac{\sin \theta_e}{\theta_e} \right) = \frac{\partial}{\partial \theta} \left( \frac{\sin \theta_e}{\theta_e} \right) \dot{\theta}_e \in \mathcal{L}_\infty \). Therefore,

\[
\dot{\omega} = \hat{\omega} + \dot{\alpha}_\omega = \frac{1}{\gamma} \hat{\theta}_e + \dot{\alpha}_\omega - c_7 \ddot{\omega} + \xi_2 \in \mathcal{L}_\infty \tag{68}
\]

By the way,

\[
\frac{d}{dt} \ddot{\omega} = 2\ddot{\hat{\omega}}
\]

\[
= 2\ddot{\hat{\omega}}(-\ddot{\alpha}_e + u_1)
\]

\[
= 2\ddot{\hat{\omega}}(\dddot{x}_e - c_5 \dddot{v} + \xi_1) \in \mathcal{L}_\infty \tag{69}
\]

Hence, with the help of Lemma 1, it follows that \( \ddot{v} \to 0, (t \to \infty) \). From the above results, we can conclude that

\[
v - v_r = (\ddot{v} + \alpha_v) - v_r
\]

\[
= \ddot{\ddot{v}} + v_r \cos \theta_e - c_3 \omega y_e + c_3 \omega (\omega x_e - v_r \sin \theta_e)
\]

\[
+ c_4 \dddot{x}_e - v_r \to 0, (t \to \infty) \tag{70}
\]

C Lemma

**Lemma 1** (Barbalat [8]) If \( \phi : \mathcal{R}_+ \to \mathcal{R} \) is uniformly continuous and if the limit of the integral \( \int_0^t \phi(\tau) \, d\tau \) exists as \( t \to \infty \) and is finite, then \( \lim_{t \to \infty} \phi(t) = 0 \).

**Lemma 2** [6] Consider a scalar system

\[
\dot{x} = -cx + p(t) \tag{71}
\]

where \( c > 0 \) and \( p(t) \) is a bounded and uniformly continuous function. If, for any initial time \( t_0 \geq 0 \) and any initial condition \( x(t_0) \), the solution \( x(t) \) is bounded and converges to 0 as \( t \to \infty \), then \( \lim_{t \to \infty} p(t) = 0 \).

D The boundedness of the original state trajectory

In this section, we show that the original state trajectory, namely, \( [\dddot{x}_e, \dddot{y}_e, \dddot{\theta}_e, \dddot{\omega}, \dot{\omega}]^T \) is bounded under the conditions of Theorem 1. From Theorem 1, \( \dddot{x}_e, \dddot{y}_e, \dddot{\theta}_e, \dddot{\omega}, \dot{\omega} \in \mathcal{L}_\infty \). Hence, \( \omega = \dddot{\omega} + \omega_r + \gamma y_e v_r \sin \theta_e / \theta_e + c_5 \gamma \dot{\theta}_e \in \mathcal{L}_\infty \) and \( x_e = \dddot{x}_e + c_3 \omega y_e \in \mathcal{L}_\infty \). Further, \( \dddot{y}_e = -c_3 \omega^2 y_e - \omega \dddot{x}_e + v_r \sin \theta_e \in \mathcal{L}_\infty \) and \( \dddot{\theta}_e = -c_5 \gamma \theta_e - \gamma y_e v_r \sin \theta_e / \theta_e - \dddot{\omega} \in \mathcal{L}_\infty \). From these facts, it can be shown that \( \dot{\alpha}_\omega \in \mathcal{L}_\infty \). Hence, \( \dddot{\omega} = \dot{\omega}_e / \gamma + \dot{\alpha}_\omega - c_7 \ddot{\omega} + d_2 \in \mathcal{L}_\infty \). Hence, \( \dddot{v} = \dddot{\ddot{v}} + c_4 \dddot{x}_e + v_r \cos \theta_e - c_3 \omega y_e + c_3 \omega (\omega x_e - v_r \sin \theta_e) \in \mathcal{L}_\infty \).
In this section, we show that the boundedness of $u(x)$ is guaranteed under the conditions of Theorem 1 and several additional conditions, namely, $\ddot{v}_r, \ddot{\omega}_r, \dddot{d}_3 \in \mathcal{L}_\infty$. Firstly, we show the boundedness of $u_1$. From Theorem 1 and Appendix D, $\dot{x}_e = \omega y_e - c_4 \bar{x}_e - \bar{v} \in \mathcal{L}_\infty$ and $\dot{x}_e = \dot{x}_e + c_3 \bar{\omega} y_e + c_2 \bar{\omega} \dot{y}_e \in \mathcal{L}_\infty$ and $\dot{\omega} = \theta_e / \gamma - c_7 \bar{\omega} - \xi_2 + d_2 \in \mathcal{L}_\infty$. Note that, from Assumption 1, $d_2 = -\tau_{d3} / J \in \mathcal{L}_\infty$. Based on these results, we can show that $\ddot{y}_e, \ddot{\theta}_e \in \mathcal{L}_\infty$. Moreover, by using the above results and the additional assumptions on the reference trajectories, namely, $\dddot{v}_r, \dddot{\omega}_r \in \mathcal{L}_\infty$, we can show that $\dddot{\alpha}_\omega \in \mathcal{L}_\infty$. Hence, $\ddot{\omega} = \dot{\theta}_e / \gamma + \dddot{\omega} - c_7 \bar{\omega} + \xi_2 + d_2 \in \mathcal{L}_\infty$. Note that we used $\dddot{d}_3 = -d_3 / J \in \mathcal{L}_\infty$. Therefore, we can show $\dot{\alpha}_v \in \mathcal{L}_\infty$. Hence, we can conclude that $\dot{u}_1 = \dddot{x}_e + \dddot{\alpha}_v - c_6 \bar{v} + \xi_1 \in \mathcal{L}_\infty$. Finally, we can show that $u_2$ is bounded, since $u_2 = \theta_e / \gamma + \dddot{\omega} - c_7 \bar{\omega} + \xi_2 \in \mathcal{L}_\infty$ from Eq. (23).
Figure 3: Tracking trajectories (dashed: reference robot, solid: real robot with $\xi(x)$, dash-dot: real robot without $\xi(x)$)

Figure 4: Tracking errors with $\xi(x)$ (solid: $x_e$, dashed: $y_e$, dash-dot: $\theta_e$)

Figure 5: Tracking errors without $\xi(x)$ (solid: $x_e$, dashed: $y_e$, dash-dot: $\theta_e$)